Exercises Module 2

Corrections

Exercise 2.1

The shear force is equal to the shear stress on the surface element multiplied by the surface area. To identify the relevant component contributing to the shear force, we can use the general formula for the shear force F_{shear}:

$$F_{shear} = -\tau_{jk}\big|_j (\hat{\boldsymbol{n}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}})\hat{\boldsymbol{k}} A_i$$

with the direction of flow k (unit vector $\hat{\boldsymbol{k}}$) and the direction of momentum transport j (unit vector $\hat{\boldsymbol{j}}$). $\hat{\boldsymbol{n}}_{\boldsymbol{i}}$ is the surface normal vector of the considered surface A_i.

From Newton's law of viscosity, we know that shear stress as a momentum flux in direction j originates from a non-zero velocity gradient $\frac{dv_k}{dj}$ of v_k in direction of j.

If fluid is flowing over the surface of a solid body, the no-slip condition has to be fulfilled at the surface. If the body is not in motion, this implies a velocity of zero right at the surface and on the whole surface. Therefore, there can be no gradient of the velocity along the directions moving on the surface, but only in the directions moving away from the surface element. For example, the velocity on the surface does not vary in z or θ in the case of a solid cylinder, because it needs to be zero everywhere on the cylinder.

With that in mind, we can derive the terms for the shear force on the indicated surface elements:

a) $\boldsymbol{dF_{shear}} = -\tau_{r\theta}|_{r=R} R d\theta \times dz \times (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{\theta}} = -\tau_{r\theta}|_{r=R} R d\theta\, dz\, \hat{\boldsymbol{\theta}}$

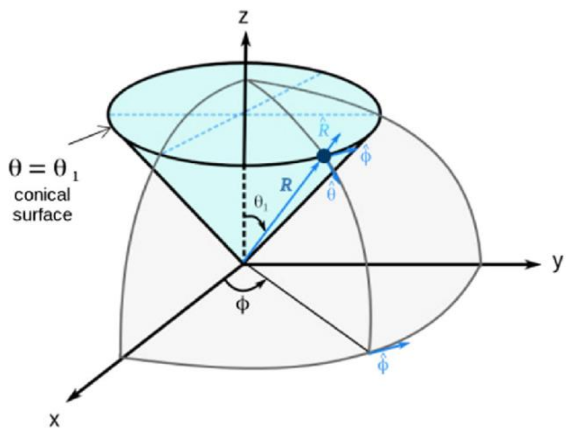
b) $\boldsymbol{dF_{shear}} = -\tau_{rz}|_{r=R} R d\theta \times dz \times (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{z}} = -\tau_{rz}|_{r=R} R d\theta\, dz\, \hat{\boldsymbol{z}}$

c) $\boldsymbol{dF_{shear}} = -\tau_{r\theta}|_{r=R} R d\theta \times R\sin\theta d\phi \times (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{\theta}} = -\tau_{r\theta}|_{r=R} R^2 \sin\theta\, d\theta d\phi\, \hat{\boldsymbol{\theta}}$

d) $\boldsymbol{dF_{shear}} = -\tau_{r\phi}\big|_{r=R} R d\theta \times R\sin\theta d\phi \times (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}})\, \hat{\boldsymbol{\phi}} = -\tau_{r\theta}|_{r=R} R^2 \sin\theta\, d\theta d\phi\, \hat{\boldsymbol{\phi}}$

e) $\boldsymbol{dF_{shear}} = -\tau_{\theta r}|_{\theta=\alpha} dr \times (r\sin\alpha\, d\phi) \times \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\theta}}\right)\hat{\boldsymbol{r}} = -\tau_{\theta r}|_{\theta=\alpha} r\sin\alpha\, dr d\phi\, \hat{\boldsymbol{r}}$

The coordinate system in case e) is a spherical coordinate system. The cone is placed within a sphere with the origin of the cone being at the origin of the sphere. θ and ϕ are chosen in a similar way as for a sphere.



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Exercise 2.2

In cylindrical coordinates: (r, θ, z)

The fluid is moving in the orthoradial direction (defined by $\boldsymbol{e_\theta}$), while the speed gradient is only radial (the speed does not depend on θ and z). Therefore, the velocity profile will have the form:

$$\boldsymbol{v} = v_\theta\,(r)\,\boldsymbol{e_\theta}$$

Moreover, inside the fluid, the pressure varies radially and vertically, but does not depend on θ (due to the cylindrical symmetry, nothing should vary along θ). Therefore, we have:

$$p = p(r, z)$$

We now apply the Navier-Stokes equation in cylindrical coordinates.

1) On r:

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + v_z\cancel{\frac{\partial v_r}{\partial z}} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,\cancel{(r v_r)}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}}\right) + \cancel{\rho g_r}$$

$$\boldsymbol{-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}}\ \ \textbf{(1)}$$

2) On θ:

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}} - \cancel{\frac{v_r v_\theta}{r}}\right) = -\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}\right) + \cancel{\rho g_\theta}$$

$$\boldsymbol{0 = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(r v_\theta)\right)}\ \ \textbf{(2)}$$

3) On z:

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_z}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) + \rho g_z$$

$$\boldsymbol{0 = -\frac{\partial p}{\partial z} - \rho g}\ \ \textbf{(3)}$$

Integrating equation (2) gives:

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(r v_\theta)\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\,(r v_\theta) = C_1$$

$$\frac{\partial}{\partial r}\,(r v_\theta) = C_1\,r$$

$$r v_\theta = C_1\frac{r^2}{2} + C_2$$

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$$

The boundary conditions are:

$v_\theta|_{r=0} = 0$ (since at r = 0 there is no motion)

$\Longrightarrow\ C_2 = 0$

and $v_\theta(R) = \Omega R$ (Known wall velocity)

$\Longrightarrow\ C_1 = 2\Omega$

Therefore:

$$\boxed{\boldsymbol{v_\theta(r) = \Omega r}}$$

Integrating equation (1) gives:

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r$$

$$p(r,z) = \frac{1}{2}\,\rho \Omega^2 r^2 + f(z)$$

We use equation (3) to determine f(z):

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{df}{dz} = -\rho g$$

$$f(z) = -\rho g z + C_3$$

Therefore, overall:

$$p(r,z) = \frac{1}{2}\,\rho \Omega^2 r^2 - \rho g z + C_3$$

Boundary condition:

$p(r, z = \delta(r)) = p_{atm}$

where $\delta(r)$ is the free surface of the liquid.

By applying this boundary condition, we can get the equation for the surface:

$$p_{atm} = \frac{1}{2}\,\rho\Omega^2 r^2 - \rho g \delta(r) + C_3$$

$$\delta(r) = \frac{1}{2g}\,\Omega^2 r^2 - \frac{p_{atm}}{\rho g} + \frac{C_3}{\rho g}$$

and the boundary condition we can apply to calculate the constant is at the lowest point of the fluid surface:

$\delta(r = 0) = \ z_0$

$\Longrightarrow p_{atm} = -\rho g z_0 + C_3 \qquad \Longrightarrow C_3 = p_{atm} + \rho g z_0$

Therefore

$$\boxed{\boldsymbol{\delta(r)} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g}}\,\boldsymbol{\Omega^2 r^2} + \boldsymbol{z_0}}$$

<u>Note</u>: If we want to calculate the pressure, we can substitute the constant we calculated into the pressure equation and finally get:

$$p(r,z) - p_{atm} = \frac{1}{2}\,\rho\Omega^2 r^2 + \rho g(z_0 - z)$$

Exercise 2.4

The fluid is flowing in the z direction and we have velocity gradients in the x and y directions. Therefore:

$$\boldsymbol{v} = v_z(x,y)\boldsymbol{e_z}$$

Moreover, ignoring any gravitational effects, the pressure is uniform on a section of the pipe and varies along the flow:

$$p = p(z)$$

The velocity profile satisfies the boundary conditions:

$$v(\pm B, y) = 0\,;\, v(x, \pm B) = 0$$

$$\frac{\partial v_z}{\partial x}(x = 0, y = 0) = 0;\ \frac{\partial v_z}{\partial y}(x = 0, y = 0)$$

First, let’s see if the proposed solution matches the boundary conditions:

$$v_z = \frac{(p_0 - p_L)B^2}{4\mu L}\left[\left(1 - \frac{y^2}{B^2}\right)\left(1 - \frac{x^2}{B^2}\right)\right]$$

$$v(\pm B, y) = 0 \ \textbf{yes}$$

$$v(0, \pm B) = 0 \ \textbf{yes}$$

$$\frac{\partial v_z}{\partial x} = -\frac{(p_0 - p_L)B^2}{4\mu L}\left(1 - \frac{y^2}{B^2}\right)\frac{2}{B^2}x \ \textbf{yes}$$

$$\frac{\partial v_z}{\partial y} = -\frac{(p_0 - p_L)B^2}{4\mu L}\left(1 - \frac{x^2}{B^2}\right)\frac{2}{B^2}y \ \textbf{yes}$$

Ok looks good so far, lets next check the momentum balance. In Cartesian coordinates, the Navier-Stokes equation projected on the x and y axis are irrelevant here (we just end up with the hydrostatic pressure):

On the z axis:

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x}\frac{\partial v_z}{\partial x} + \cancel{v_y}\frac{\partial v_z}{\partial y} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) + \cancel{\rho g_z}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right)$$

This relation must be true over the entire domain of our square duct. Let’s check this. Solving for the second derivatives using the given velocity profile:

$$\frac{\partial^2 v_z}{\partial x^2} = -\frac{(p_0 - p_L)}{2\mu L}\left(1 - \frac{y^2}{B^2}\right)$$

$$\frac{\partial^2 v_z}{\partial y^2} = -\frac{(p_0 - p_L)}{2\mu L}\left(1 - \frac{x^2}{B^2}\right)$$

Plugging in the Navier-Stokes equation:

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) \Rightarrow -\frac{\partial p}{\partial z} + \mu\left(-\frac{(p_0 - p_L)}{2\mu L}\left(1 - \frac{y^2}{B^2}\right) - \frac{(p_0 - p_L)B^2}{2\mu L}\left(1 - \frac{x^2}{B^2}\right)\right)$$

$$\Rightarrow -\frac{\partial p}{\partial z} - \frac{(p_0 - p_L)}{2L}\left(\left(1 - \frac{y^2}{B^2}\right) + \left(1 - \frac{x^2}{B^2}\right)\right) = 0$$

We have the sum of three independent functions of z, y, x. If the Navier Stokes equation was verified, the sum of these functions should be always equal to 0.

In particular, on the current line defined by x=0 and y=0 (center of the pipe), in analogy to flow in a circular pipe the pressure drop must be linear so we must have:

$$\frac{\partial p}{\partial z} = -\frac{(p_0 - p_L)}{L}$$

And since *p* depends only on *z*, this must therefore be true in all the fluid. Which implies that:

$$\frac{(p_0 - p_L)}{2L}\left[2 - \left(1 - \frac{y^2}{B^2}\right) - \left(1 - \frac{x^2}{B^2}\right)\right] = 0$$

For any x and y on the section we are considering.

Clearly this is not true as for non-zero values of x and y the equation is not correct. Therefore the momentum balance is not satisfied and thus velocity profile we are given is wrong.

For those that are curious the correct velocity profile is given by the following (rapidly converging) infinite series:

$$v_z(x,y) = \frac{1}{\mu}\frac{dp}{dz}\left[\frac{1}{2}(y^2 - B^2) - 16B^2\sum_{n=1}^{\infty}\frac{(-1)^n}{u_n^3}\frac{\cosh\left(u_n\frac{x}{2B}\right)}{\cosh\left(u_n\frac{1}{2B}\right)}\cosh\left(u_n\frac{y}{2B}\right)\right]$$

where $u_n = (2n-1)\pi$. Yes, this is the real solution ☺. In another example we will show you how these infinite series arise.

Exercise 2.5

In cylindrical coordinates. Due to the symmetry of the problem, nothing should be dependent on θ. Moreover, the flow is only radial.

Therefore, we have:

$$\boldsymbol{v} = v_r(r,z)\boldsymbol{e_r}$$

And

$$p = p(r,z)$$

The boundary conditions are:

$$v_r(r, \pm b) = 0; \quad \frac{\partial v_r}{\partial z}(r,0) = 0$$

The projected Navier Stokes equations give, after simplification:

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial z} = 0 \text{ (neglecting gravity)}$$

Moreover, the continuity equation is always valid:

$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{\nabla} \cdot \rho \boldsymbol{v})$$

In cylindrical coordinates the continuity equation is :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Since we are at steady state, and $\boldsymbol{v} = v_r(r,z)\boldsymbol{e_r}$ the continuity equation becomes (assuming constant density):

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r v_r) = 0$$

Integrating the simplified continuity equation (keeping in mind that $v_r(r,z)$):

$$r v_r = f(z) \Leftrightarrow v_r = \frac{f(z)}{r}$$

Replacing this result into the simplified Navier Stokes:

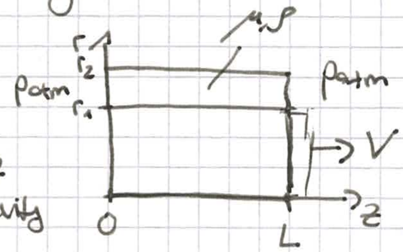
$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{dp}{dr} + \mu \frac{\partial^2 v_r}{\partial z^2}$$

$$\boxed{-\rho \frac{f(z)^2}{r^3} = -\frac{dp}{dr} + \frac{\mu}{r}\frac{d^2 f}{dz^2}}$$

We do not know how to solve such an equation, unless the right term is negligible (see next module).

Exercise 2.3 – wire coating

constants : r_1, r_2, L, V



a) in annular space : $r_1 \leq r \leq r_2$
steady state, neglecting gravity

$v = v_z(r) \qquad v_\theta = 0 \qquad v_r = 0$

Boundary conditions: $\quad v_z(r_1) = V$
$\qquad v_z(r_2) = 0$

Navier – Stokes equations:

(I) $\quad 0 = 0$

(II) $\quad 0 = 0$

(III) $\quad 0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_z}{\partial r}\right)\right]$

$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_z(r)}{\partial r}\right)$ $\qquad$ known: $\frac{\partial p}{\partial z} = 0$ from p_{atm}

$0 = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_z(r)}{\partial r}\right)$

$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_z}{\partial r}\right)$

$0 = \frac{1}{r}\left(1 \cdot \frac{\partial v_z}{\partial r} + r \cdot \frac{\partial^2 v_z}{\partial r^2}\right)$

$-\frac{1}{r}\frac{\partial v_z}{\partial r} = \frac{\partial^2 v_z}{\partial r^2}$ $\qquad$ Substitute: $\frac{\partial v_z}{\partial r} = u(r)$

$-\frac{1}{r}u(r) = \frac{du(r)}{dr}$

$\int -\frac{1}{r}\,dr = \int \frac{1}{u(r)}\,du(r)$

$-\ln(r) + C_1 = \ln(u(r)) = \ln\left(\frac{dv_z(r)}{dr}\right)$

$e^{-\ln(r) + C_1} = \frac{dv_z(r)}{dr}$

$v_z(r) = \int e^{-\ln(r) + C_1}\,dr = \int \frac{1}{r} \cdot C_1'\,dr$

$v_z(r) = C_1' \cdot \ln(r) + C_2$

1\. BC: $\quad v_z(r_1) = C_1' \cdot \ln(r_1) + C_2 = V$ $\qquad$ 2. BC: $v_z(r_2) = C_1'\ln(r_2) + C_2 = 0$
$\qquad C_2 = -C_1\ln(r_2)$

$\Rightarrow \quad V = C_1'\ln(r_1) - C_1\ln(r_2)$

$C_1 = \frac{V}{\ln(r_1/r_2)} \qquad C_2 = -\frac{V}{\ln(r_1/r_2)} \cdot \ln(r_2)$

$v_z(r) = \frac{V}{\ln(r_1/r_2)} \cdot \ln(r) - \frac{V}{\ln(r_1/r_2)} \cdot \ln(r_2) = \frac{V}{\ln(r_1/r_2)} \cdot \left(\ln\left(\frac{r}{r_2}\right)\right)$

b)

$$\dot{m}_z = \int \rho (\vec{v} \cdot \vec{n})\, dA$$

$$= \int_{r_1}^{r_2} \rho \cdot v_z(r)\, dA$$

$$= \int_{r_1}^{r_2} \rho \cdot v_z(r)\, 2\pi r\, dr$$

$$= \rho\, 2\pi \frac{V}{\ln(\frac{r_1}{r_2})} \int_{r_1}^{r_2} \ln\left(\frac{r}{r_2}\right) \cdot r\, dr$$

$$= \frac{2\pi \rho V}{\ln(r_1/r_2)} \left[\int_{r_1}^{r_2} r \cdot \ln(r)\, dr - \int_{r_1}^{r_2} \ln(r_2)\, r\, dr \right]$$

$$= \frac{2\pi \rho V}{\ln(r_1/r_2)} \left[\frac{1}{2} r^2 \ln(r) \Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{1}{r} \cdot \frac{1}{2} r^2\, dr - \frac{1}{2} \ln(r_2)\, r^2 \Big|_{r_1}^{r_2} \right]$$

$$\dot{m}_z = \frac{2\pi \rho V}{\ln(r_1/r_2)} \left[\frac{1}{2} r_1^2 \ln(r_2/r_1) + \frac{1}{4}(r_1^2 - r_2^2) \right]$$

c) $$\lim_{r_1 \to 0} \dot{m}_z = \lim_{r_1 \to 0} \frac{2\pi \rho V}{\ln(\frac{r_1}{r_2})} \left(\frac{r_1^2}{2} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{4}(r_1^2 - r_2^2) \right]$$

$$= 2\pi \rho V \cdot \Big[\lim_{r_1 \to 0} \Big(\frac{r_1^2}{2} + \frac{1}{4} \Big(\underbrace{\frac{r_1^2 - r_2^2}{\ln(r_1/r_2)}}_{\to -\infty} \Big)$$

$$= 0$$

d) Thickness at p_{atm}

Approach 1:

$V_z(r) = V = const$, mass flow rate must be constant

$$\dot{m}_z = \dot{m}_{out}$$

$$\dot{m}_{out} = \int_{r_1}^{x} \rho\, v_z(r)\, dA \qquad x = r_1 + \delta$$

$$= \int_{r_1}^{x} \rho\, V\, 2\pi r\, dr$$

$$= 2\pi \rho V \int_{r_1}^{x} r\, dr$$

$$\dot{m}_{out} = 2\pi \rho V (x^2 - r_1^2)$$

$$\dot{m}_{out} = \dot{m}_z$$

$$2\pi \rho V (x^2 - r_1^2) = \frac{2\pi \rho V}{\ln(r_1/r_2)} \left[\frac{r_1^2}{2} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{4}(r_1^2 - r_2^2) \right]$$

$$x^2 = \frac{2}{\ln(r_1/r_2)} \left[\frac{r_1^2}{2} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{4}(r_1^2 - r_2^2) \right] + r_1^2$$

$$x^2 = \frac{1}{\ln(r_1/r_2)} \left(r_1^2 \cdot \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2 \right) + r_1^2$$

$$x = \sqrt{\frac{2}{\ln(r_1/r_2)}\left[\frac{r_1^2}{2}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{4}(r_1^2 - r_2^2)\right] + r_1^2} = r_1 + \delta$$

$$\delta = x - r_1$$

$$\delta = \sqrt{\frac{2}{\ln(r_1/r_2)}\left[\frac{r_1^2}{2}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{4}(r_1^2 - r_2^2)\right] + r_1^2} - r_1$$

Approach 2: surface area calculated from ratio of volumetric flow rate $\dot{V}$ and velocity

$$\dot{V} = \frac{\dot{m}}{\rho}$$

$$\frac{\dot{m}/\rho}{V} = \pi(r_1 + \delta^2) - \pi r_1^2$$

$$\frac{\dot{m}}{\rho \cdot V} = 2\pi r_1 \delta + \pi \delta^2$$

$$0 = \pi\delta^2 + 2\pi r_1 \delta - \frac{\dot{m}}{\rho V}$$

$$0 = \delta^2 + 2 r_1 \delta - \frac{\dot{m}}{\pi \cdot V \cdot \rho}$$

$$\delta_+ = -\frac{2r_1}{2} + \sqrt{\left(\frac{2r_1}{2}\right)^2 + \frac{\dot{m}}{\pi V \rho}}$$

$$= -r_1 + \sqrt{r_1^2 + \frac{2}{\ln(r_1/r_2)}\left(\frac{1}{2}r_1^2(\ln(r_2/r_1) + \frac{1}{4}(r_1^2 - r_2^2)\right]}$$

$$\delta_+ = \sqrt{r_1^2 + \frac{1}{\ln(r_1/r_2)}\left(r_1^2\ln(r_2/r_1) + \frac{1}{2}r_1^2 - \frac{1}{2}r_2^2\right)} - r_1$$

e) Force

$$-F_{surf} = F_{shear} = \iint_A -\tau_{rz}\big|_{r=r_1} (\hat{n}_r \cdot \hat{r}) \cdot \hat{z}\, dA$$

$$= \int_0^L\int_0^{2\pi} -\tau_{rz}\big|_{r=r_1} (1) \cdot \hat{z}\; r_1 \cdot d\theta\, dz \qquad \tau_{rz} = -\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$= \int_0^L\int_0^{2\pi} \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\bigg|_{r=r_1} \hat{z}\, r_1\, d\theta\, dz$$

$$= \mu \cdot \frac{d v_z}{dr}\bigg|_{r=r_1} \hat{z}\; r_1 \cdot 2\pi L$$

$$-F_{surf} = F_{rz} = \mu \cdot C_1 \cdot \frac{1}{r_1}\hat{z}\; r_1 2\pi L = \frac{2\pi\mu L V}{\ln(r_1/r_2)}\hat{z}$$